

COHERENT CONFIGURATIONS ASSOCIATED WITH TI-SUBGROUPS

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ABSTRACT. Let \mathcal{X} be a coherent configuration associated with a transitive group G . In terms of the intersection numbers of \mathcal{X} , a necessary condition for the point stabilizer of G to be a TI-subgroup, is established. Furthermore, under this condition, \mathcal{X} is determined up to isomorphism by the intersection numbers. It is also proved that asymptotically, this condition is also sufficient. More precisely, an arbitrary homogeneous coherent configuration satisfying this condition is associated with a transitive group, the point stabilizer of which is a TI-subgroup. As a byproduct of the developed theory, recent results on pseudocyclic and quasi-thin association schemes are generalized and improved. In particular, it is shown that any scheme of prime degree p and valency k is associated with a transitive group, whenever $p > 1 + 6k(k-1)^2$.

Keywords: coherent configuration, permutation group, TI-subgroup

1. INTRODUCTION

Recall that a subgroup H of a finite group G is said to be TI (trivial intersection) if $H^g \cap H = H$ or 1 for all $g \in G$. In particular, this condition is obviously satisfied if H is of prime order or G is a Frobenius group and H is a Frobenius complement. Some other examples can be found, e.g., in [2, 7, 13]. One of the goals of this paper is to prove that, at least asymptotically, the concept of TI-subgroup can be defined in a purely combinatorial way (see Theorem 1.2 below). This enables us to generalize and improve recent results on pseudocyclic and quasi-thin association schemes [10, 11].

Let $G \leq \text{Sym}(\Omega)$ be a transitive permutation group and H the one point stabilizer of G . Suppose that H is a TI-subgroup of G . Then from Theorem 3.1 below, it follows that

$$(1) \quad |X| \in \{1, k\}, \quad X \in \text{Orb}(G, \Omega),$$

where k is the order of H . Moreover, set c to be the maximum number of points fixed by the permutations belonging to a right H -coset of G other than H . Then by Lemma 3.4, we have

$$(2) \quad c \leq mk,$$

where m is the index of H in $N_G(H)$. In Section 3, it is shown that this bound is tight, but in general conditions (1) and (2) do not guarantee that H is a TI-subgroup of G . Nevertheless, the situation changes if the ratio n/m with $n = |\Omega|$ is enough large in comparison with k (Proposition 3.6).

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What we said in the above paragraph has a clear combinatorial meaning. Namely, let $\mathcal{X} = (\Omega, S)$ be the (homogeneous) coherent configuration associated with the group G (the exact definitions concerning coherent configurations can be found in Section 2). We say that \mathcal{X} is a *TI-scheme* if H is a TI-subgroup of G . In this case, condition (1) means that

$$(3) \quad S = S_1 \cup S_k,$$

where S_1 and S_k are the sets of basis relations of \mathcal{X} that have valences 1 and k , respectively.¹ The numbers c , m , and k in condition (2) are, respectively, the indistinguishing number², the cardinality of S_1 , and the maximal valency of \mathcal{X} . Thus, both conditions (1) and (2) can be expressed in terms of \mathcal{X} without mentioning G .

Definition 1.1. *A homogeneous coherent configuration \mathcal{X} satisfying conditions (3) and (2) is said to be a pseudo-TI scheme of valency k and index n/m .*

The above discussion shows that every TI-scheme is also pseudo-TI. Furthermore, the result [10, Theorem 3.2] shows that every pseudocyclic scheme is pseudo-TI with $m \in \{1, n\}$ and $c = k - 1$. On the other hand, many quasi-thin schemes are also pseudo-TI but this time with $k = 2$ (Theorem 3.9). Some other examples of pseudo-TI schemes (in particular, those which are not TI) can be found in Section 3.

As was mentioned, the main result of the present paper shows that asymptotically, every pseudo-TI scheme \mathcal{X} is also TI. However, we can say more. Namely, in this case, \mathcal{X} is determined up to isomorphism by its intersection numbers, i.e., is *separable* in terms of [3]. Thus, at least asymptotically, there is a 1-1 correspondence between the permutation groups with TI point stabilizers and pseudo-TI schemes.

Theorem 1.2. *Every pseudo-TI scheme of valency k and index greater than $1 + 6k^2(k - 1)$, is a separable TI-scheme.*

The bound in Theorem 1.2 is not tight. To see this, let \mathcal{X} be a pseudo-TI scheme with $k = 2$. Then \mathcal{X} is a quasi-thin association scheme. Assume that the index of \mathcal{X} is equal to 2 (such schemes do exist). Then from [11, Theorem 1.1], it follows that \mathcal{X} is separable and *schurian*, i.e., is associated with an appropriate permutation group. Thus, this scheme is TI. However, in this case the bound given in Theorem 1.2 is not achieved.

Formula (3) shows that $|S| = m + (n - m)/k$. Therefore Theorem 1.2 immediately follows from inequality (2) and the theorem below, in which the numbers k , c , and m are called the *parameters* of the pseudo-TI scheme \mathcal{X} .

Theorem 1.3. *Let $\mathcal{X} = (\Omega, S)$ be a pseudo-TI scheme with parameters k , c , and m . Suppose that*

$$|S| > m + 6c(k - 1),$$

Then \mathcal{X} is schurian and separable.

It should be mentioned that according to [10, Theorems 1.1 and 1.3], the statement of Theorem 1.3 is true if the scheme \mathcal{X} is pseudocyclic and its rank $|S|$ is greater than ak^4 for some $a > 0$. In this case, \mathcal{X} must be a Frobenius scheme, i.e., \mathcal{X} is associated with a Frobenius group, the point stabilizer of which coincides with

¹In the terminology of [14], the set S_1 is the thin radical of \mathcal{X} .

²In the complete colored graph representing \mathcal{X} , c is the maximum number of triangles with fixed base, the other two sides of which are monochrome arcs (see also Subsection 2.1).

the Frobenius complement. Moreover, since here $c = k - 1$, we can substantially improve the above bound as follows.

Corollary 1.4. *Any pseudocyclic scheme of valency $k > 1$ and rank greater than $1 + 6(k - 1)^2$ is a Frobenius scheme.*

The brilliant Hanaki-Uno theorem on association schemes of prime degree p states that such a scheme must be pseudocyclic [6]. In particular, formula (3) holds for some k called the valency of the scheme. Thus, as an immediate consequence of Theorem 1.3, we obtain the following statement.

Corollary 1.5. *Any scheme of prime degree p and valency k is schurian, whenever $p > 1 + 6k(k - 1)^2$.*

To make the paper possibly self-contained, we cite the basics of coherent configurations in Section 2. In Section 3, we study TI- and pseudo-TI coherent configurations in detail; our presentation here includes also several examples. The proof of Theorem 1.3 is based on a theory of matching configurations, which are introduced and studied in Section 4. This theory is used in Section 5, where we prove a quite general Theorem 5.1, which shows that the one point extension of a homogeneous coherent configuration with sufficiently many relations of maximum valency must contain a big matching subconfiguration. These results are the main ingredients in our proof of Theorem 1.3 given in Section 6.

Notation. Throughout the paper, Ω denotes a finite set.

The diagonal of the Cartesian product $\Omega \times \Omega$ is denoted by 1_Ω ; for any $\alpha \in \Omega$ we set $1_\alpha = 1_{\{\alpha\}}$.

For $r \subseteq \Omega \times \Omega$, set $r^* = \{(\beta, \alpha) : (\alpha, \beta) \in r\}$ and $\alpha r = \{\beta \in \Omega : (\alpha, \beta) \in r\}$ for all $\alpha \in \Omega$.

For $s \subseteq \Omega \times \Omega$ we set $r \cdot s = \{(\alpha, \gamma) : (\alpha, \beta) \in r, (\beta, \gamma) \in s \text{ for some } \beta \in \Omega\}$. If S and T are sets of relations, we set $S \cdot T = \{s \cdot t : s \in S, t \in T\}$.

For $S \in 2^{\Omega^2}$, set $S^\# = S \setminus \{\emptyset\}$, denote by S^\cup the set of all unions of the elements of S , and put $S^* = \{s^* : s \in S\}$ and $\alpha S = \cup_{s \in S} \alpha s$, where $\alpha \in \Omega$.

2. COHERENT CONFIGURATIONS

This section contains well-known basic facts on coherent configurations. All of them can be found in [3] and papers cited there.

2.1. Main definitions. Let Ω be a finite set, and let S be a partition of a subset of $\Omega \times \Omega$. Then the pair $\mathcal{X} = (\Omega, S)$ is called a *partial coherent configuration* on Ω if the following conditions hold:

- (C1) $1_\Omega \in S^\cup$,
- (C2) $S^* = S$,
- (C3) given $r, s, t \in S$, the number $c_{rs}^t = |\alpha r \cap \beta s^*|$ does not depend on the choice of $(\alpha, \beta) \in t$.

If S is a partition of $\Omega \times \Omega$, then \mathcal{X} is a *coherent configuration* in the usual sense. The elements of Ω and S , and the numbers c_{rs}^t are called the *points* and *basis relations*, and the *intersection numbers* of \mathcal{X} , respectively. The numbers $|\Omega|$ and $|S|$ are called the *degree* and the *rank* of \mathcal{X} . The basis relation containing a pair $(\alpha, \beta) \in \Omega \times \Omega$ is denoted by $r(\alpha, \beta)$. For $r, s \in S^\cup$, the set of basic relations contained in $r \cdot s$ is denoted by rsv .

Denote by $\Phi = \Phi(\mathcal{X})$ the set of $\Lambda \subseteq \Omega$ such that $1_\Lambda \in S$. The elements of Φ are called the *fibers* of \mathcal{X} . In view of condition (C1), the set Ω is the disjoint union of all of them. Moreover, for each $r \in S$, there exist uniquely determined fibers Λ and Δ such that $r \subseteq \Lambda \times \Delta$. Note that the number $|\alpha r| = c_{rr^*}^t$ with $t = 1_\Lambda$, does not depend on $\alpha \in \Lambda$. This number is called the *valency* of r and denoted n_r . One can see that if $\Delta \in \Phi^\cup$ and S_Δ is the set of all basis relations contained in $\Delta \times \Delta$, then

$$\mathcal{X}_\Delta = (\Delta, S_\Delta)$$

is a partian coherent configuration, called the *restriction* of \mathcal{X} to Δ .

2.2. Homogeneous coherent configurations. A coherent configuration \mathcal{X} is said to be *homogeneous* or a *scheme* if $1_\Omega \in S$. In this case, $n_r = n_{r^*} = |\alpha r|$ for all $r \in S$ and $\alpha \in \Omega$. Given a positive integer k , we set

$$(4) \quad S_k = \{r \in S : n_r = k\}.$$

One can see that the set S_1 forms a group with respect to the relation product. In what follows, we will also use the following well-known identities for the intersection numbers of a homogeneous coherent configurations:

$$(5) \quad n_r n_s = \sum_{t \in S} n_t c_{rs}^t, \quad c_{r^* s^*}^{t^*} = c_{sr}^t, \quad n_t c_{rs}^{t^*} = n_r c_{st}^{r^*} = n_s c_{tr}^{s^*}$$

for all $r, s, t \in S$.

The *indistinguishing number* $c(r)$ of a relation $r \in S$ is defined to be the sum of the intersection numbers $c_{ss^*}^r$ with $s \in S$ (see [10]). For each pair $(\alpha, \beta) \in r$, we obviously have

$$(6) \quad c(r) = |\Omega_{\alpha, \beta}|, \quad \text{where} \quad \Omega_{\alpha, \beta} = \{\gamma \in \Omega : r(\gamma, \alpha) = r(\gamma, \beta)\}.$$

The maximum $c(\mathcal{X})$ of the numbers $c(r)$, $r \in S^\#$, is called the *indistinguishing number* of the coherent configuration \mathcal{X} .

2.3. Point extensions. There is a natural partial order \leq on the set of all coherent configurations on the same set. Namely, given two coherent configurations $\mathcal{X} = (\Omega, S)$ and $\mathcal{X}' = (\Omega, S')$, we set

$$\mathcal{X} \leq \mathcal{X}' \Leftrightarrow S^\cup \subseteq (S')^\cup.$$

The minimal and maximal elements with respect to this ordering are, respectively, the *trivial* and *complete* coherent configurations: the basis relations of the former one are the reflexive relation 1_Ω and (if $n > 1$) its complement in $\Omega \times \Omega$, whereas the basis relations of the latter one are all singletons.

Given two coherent configurations \mathcal{X}_1 and \mathcal{X}_2 on Ω , there is a uniquely determined coherent configuration $\mathcal{X}_1 \cap \mathcal{X}_2$ also on Ω , the relation set of which is $(S_1)^\cup \cap (S_2)^\cup$, where S_i is the set of basis relations of \mathcal{X}_i , $i = 1, 2$. This enables us to define the *coherent closure* of a set R of binary relations on Ω by

$$(\Omega, \overline{R}) = \bigcap_{T: R \subseteq T^\cup} (\Omega, T),$$

where T runs over the partitions of $\Omega \times \Omega$ such that (Ω, T) is a coherent configuration. Note that if (Ω, R) is a coherent configuration, then $\overline{R} = R$.

Let $\mathcal{X} = (\Omega, S)$ be a coherent configuration. For the points α, β, \dots , denote by $\mathcal{X}_{\alpha, \beta, \dots}$ the coherent closure of the set $R = S \cup \{1_\alpha, 1_\beta, \dots\}$. This coherent

configuration is called the *point extension* of \mathcal{X} with respect to the points α, β, \dots . In other words, $\mathcal{X}_{\alpha, \beta, \dots}$ is the smallest coherent configuration on Ω that is larger than or equal to \mathcal{X} and has the singletons $\{\alpha\}, \{\beta\}, \dots$ as fibers. The following lemma immediately follows from the definitions.

Lemma 2.1. *Let Φ_α and S_α be, respectively, the sets of fibers and basis relations of the coherent configuration \mathcal{X}_α , where $\alpha \in \Omega$. Then*

$$\alpha r \in (\Phi_\alpha)^\cup \quad \text{and} \quad r_{x,y} \in (S_\alpha)^\cup, \quad r, x, y \in S,$$

where $r_{x,y} = r \cap (\alpha x \times \alpha y)$. Moreover, $|\beta r_{x,y}| = c_{ry}^{x*}$ for all $\beta \in \alpha x$.

In what follows, we often use the notation $r_{x,y}$ defined in Lemma 2.1. Note here that $(r_{x,y})^* = (r^*)_{y,x}$ for all r, x, y .

2.4. Direct sum and tensor product. Let $\mathcal{X} = (\Omega, S)$ and $\mathcal{X}' = (\Omega', S')$ be coherent configurations. Denote by $\Omega \sqcup \Omega'$ the disjoint union of Ω and Ω' , and by $S \boxplus S'$ the union of the set $S \sqcup S'$ and the set of all relations $\Delta \times \Delta'$ and $\Delta' \times \Delta$ with $\Delta \in \Phi(\mathcal{X})$ and $\Delta' \in \Phi(\mathcal{X}')$. Then the pair

$$\mathcal{X} \boxplus \mathcal{X}' = (\Omega \sqcup \Omega', S \boxplus S')$$

is a coherent configuration called the *direct sum* of \mathcal{X} and \mathcal{X}' . One can see that $\mathcal{X} \boxplus \mathcal{X}'$ equals the coherent closure of the set $S \sqcup S'$.

Set $S \otimes S' = \{u \otimes u' : u \in S, u' \in S'\}$, where $u \otimes u'$ is the relation on $\Omega \times \Omega'$ consisting of all pairs $((\alpha, \alpha'), (\beta, \beta'))$ with $(\alpha, \beta) \in u$ and $(\alpha', \beta') \in u'$. Then the pair

$$\mathcal{X} \otimes \mathcal{X}' = (\Omega \times \Omega', S \otimes S')$$

is a coherent configuration called the *tensor product* of \mathcal{X} and \mathcal{X}' . One can see that $\mathcal{X} \otimes \mathcal{X}'$ equals the coherent closure of the set $S \otimes 1_{\Omega'} \cup 1_\Omega \otimes S'$.

2.5. Isomorphisms and schurity. Two partial coherent configurations are called *isomorphic* if there exists a bijection between their point sets that induces a bijection between their sets of basis relations. Each such bijection is called an *isomorphism* between these two configurations. The group of all isomorphisms of a partial coherent configuration $\mathcal{X} = (\Omega, S)$ to itself contains a normal subgroup

$$\text{Aut}(\mathcal{X}) = \{f \in \text{Sym}(\Omega) : s^f = s, s \in S\}$$

called the *automorphism group* of \mathcal{X} , where $s^f = \{(\alpha^f, \beta^f) : (\alpha, \beta) \in s\}$.

Conversely, let $G \leq \text{Sym}(\Omega)$ be a permutation group, and let S be a set of orbits of the component-wise action of G on $\Omega \times \Omega$. Assume that S satisfies conditions (C1) and (C2). Then, $\mathcal{X} = (\Omega, S)$ is a partial coherent configuration; we say that \mathcal{X} is *associated* with G . A partial coherent configuration on Ω is said to be *schurian* if it is associated with some permutation group on Ω . It is easily seen that a partial coherent configuration \mathcal{X} is schurian if and only if it is the partial coherent configuration associated with the group $\text{Aut}(\mathcal{X})$.

2.6. Algebraic isomorphisms and separability. Let $\mathcal{X} = (\Omega, S)$ and $\mathcal{X}' = (\Omega', S')$ be partial coherent configurations. A bijection $\varphi : S \rightarrow S', r \mapsto r'$ is an *algebraic isomorphism* from \mathcal{X} onto \mathcal{X}' if

$$(7) \quad c_{rs}^t = c_{r's'}^{t'}, \quad r, s, t \in S.$$

In this case, \mathcal{X} and \mathcal{X}' are said to be *algebraically isomorphic*. Each isomorphism f from \mathcal{X} onto \mathcal{X}' induces an algebraic isomorphism $\varphi_f : r \mapsto r^f$ between these configurations. The set of all isomorphisms inducing the algebraic isomorphism φ is denoted by $\text{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$. In particular,

$$(8) \quad \text{Iso}(\mathcal{X}, \mathcal{X}, \text{id}_S) = \text{Aut}(\mathcal{X}),$$

where id_S is the identity mapping on S . A partial coherent configuration \mathcal{X} is said to be *separable* if for any algebraic isomorphism $\varphi : \mathcal{X} \rightarrow \mathcal{X}'$, the set $\text{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$ is not empty.

An algebraic isomorphism φ induces a bijection from S^\cup onto $(S')^\cup$: the union $r \cup s \cup \dots$ of basis relations of \mathcal{X} is taken to $r' \cup s' \cup \dots$. This bijection is also denoted by φ . One can see that φ preserves the reflexive basis relations. This extends φ to a bijection $\Phi(\mathcal{X}) \rightarrow \Phi(\mathcal{X}')$ so that $(1_\Delta)' = 1_{\Delta'}$.

In the above notation, assume that $\mathcal{Y} \geq \mathcal{X}$ and $\mathcal{Y}' \geq \mathcal{X}'$ are partial coherent configurations. We say that the algebraic isomorphism $\psi : \mathcal{Y} \rightarrow \mathcal{Y}'$ *extends* φ if $r^\psi = r^\varphi$ for all $r \in S$. In this case, we obviously have

$$\text{Iso}(\mathcal{Y}, \mathcal{Y}', \psi) \subseteq \text{Iso}(\mathcal{X}, \mathcal{X}', \varphi).$$

This immediately proves the following statement (take $\mathcal{Y} = \mathcal{X}_\alpha$).

Lemma 2.2. *Let \mathcal{X} be a coherent configuration and $\alpha \in \Omega$. Assume that for every algebraic isomorphism $\varphi : \mathcal{X} \rightarrow \mathcal{X}'$, there exists an algebraic isomorphism $\varphi_{\alpha, \alpha'} : \mathcal{X}_\alpha \rightarrow \mathcal{X}'_{\alpha'}$ with $\alpha' \in \Omega'$ that extends φ . Then \mathcal{X} is separable if so is \mathcal{X}_α .*

2.7. 1-regular coherent configurations. A point $\alpha \in \Omega$ of the coherent configuration \mathcal{X} is called *regular* if

$$|\alpha r| \leq 1 \quad \text{for all } r \in S.$$

Obviously, every point of a fiber $\Delta \in \Phi(X)$ is regular, whenever Δ contains at least one regular point. Therefore, the set of all regular points of \mathcal{X} is the union of fibers. If this set is not empty, then the coherent configuration \mathcal{X} is said to be *1-regular*; we say that \mathcal{X} is *semiregular* if each of its points is regular. Note that in the homogeneous case, a coherent configuration is 1-regular if and only if it is a thin scheme in the sense of [14]. The following statement can be found in [3, Theorem 9.3].

Theorem 2.3. *Every 1-regular coherent configuration is schurian and separable. \square*

It is easily seen that a coherent configuration is semiregular if and only if every basis relation of it is a *matching*, i.e., a binary relation of the form

$$r = \{(\alpha, f(\alpha)) : \alpha \in \Delta\}, \quad \text{where } f : \Delta \rightarrow \Delta' \text{ is a bijection.}$$

Clearly, $|r| = |\Delta| = |\Delta'|$. Note that if r' is a matching with respect to a bijection $f' : \Delta' \rightarrow \Delta''$, then $r \cdot r'$ is a matching with respect to the composition $f \circ f'$.

3. TI- AND PSEUDO-TI SCHEMES

3.1. Parameters. Throughout this section, $G \leq \text{Sym}(\Omega)$ is a transitive group and $H = G_\alpha$ is the stabilizer of the point α in G . The following statement gives a necessary and sufficient condition for H to be a TI-subgroup of G .

Theorem 3.1. *Let $\mathcal{X} = (\Omega, S)$ be the coherent configuration associated with the group G , and let k be the maximum of n_s , $s \in S$. Then H is a TI-subgroup of G if and only if $S = S_1 \cup S_k$ and H acts semiregularly on αS_k . In particular, $k = |H|$, whenever H is a TI-subgroup of G .*

Proof. By the transitivity of G , for every $\beta \in \Omega$ there exists $g \in G$ such that $\beta = \alpha^g$. It follows that

$$(9) \quad H_\beta = G_{\alpha, \beta} = G_\alpha \cap G_\beta = H \cap H^g.$$

Moreover, it is easily seen that $\beta \in \alpha S_1$ if and only if $G_\alpha = G_\beta = H^g$. Thus,

$$(10) \quad \beta \in \alpha S_1 \quad \Longleftrightarrow \quad H = H^g.$$

Assume that H is a TI-subgroup of G . Then from (9) and (10), it follows that $H_\beta = 1$ for all β belonging to $\Omega' = \Omega \setminus \alpha S_1$. Therefore, H acts semiregularly on Ω' . Since

$$\text{Orb}(H, \Omega') = \{\alpha s : s \in S \setminus S_1\},$$

this implies that $n_s = |\alpha s| = |H|$ for all $s \in S \setminus S_1$. It follows that $|H| = k$ and hence $S \setminus S_1 = S_k$ and $\Omega' = \alpha S_k$. This proves the “only if” part. Conversely, assume that $S = S_1 \cup S_k$ and H acts semiregularly on αS_k . Let $g \in G$ be such that $H \neq H^g$. Then from (10) with $\beta = \alpha^g$, it follows that $\beta \in \alpha S_k$. Therefore, $H_\beta = 1$ and hence $H \cap H^g = 1$ in view of (9). Thus, H is a TI-group, as required. \square

From now on, we assume that H is a TI-subgroup of G , i.e., the coherent configuration \mathcal{X} is a TI-scheme. Note that the group G is not uniquely determined by \mathcal{X} as the following example shows (it also shows that, in general, $G \neq \text{Aut}(\mathcal{X})$).

Example 3.2. Let $C = \mathbb{Z}_{p^2}$, $H \leq \text{Aut}(C)$, and G the permutation group induced by the action of $C \rtimes H$ on $\Omega = C$, in which C acts by right multiplications and H acts naturally. If now $|H| = p$, then the coherent configuration \mathcal{X} is a TI-scheme and $\text{Aut}(\mathcal{X}) \cong \mathbb{Z}_p \wr \mathbb{Z}_p$. One can see that G coincides with its normalizer in $\text{Aut}(\mathcal{X})$. Therefore, there are p^{p-2} conjugates of G in $\text{Aut}(\mathcal{X})$. The coherent configurations associated with them are equal to \mathcal{X} .

However, from Theorem 3.1, it follows that the orders of the groups G and H are uniquely determined by the coherent configuration \mathcal{X} , namely, $|G| = nk$ and $|H| = k$, where $n = |\Omega|$. Of course, \mathcal{X} also determines the rank $r = |S|$ of the permutation group G and the order of the group $N = N_G(H)$, where the latter immediately follows from (10) showing that $|N : H| = m$ with $m = |S_1|$. Note that

$$n = m + (r - m)k,$$

because $S = S_1 \cup S_k$ by Theorem 3.1. Nevertheless, not every schurian homogeneous coherent configuration with two distinct valences of basis relations is a TI-scheme.

Example 3.3. Let \mathcal{X} be the tensor product of a regular scheme on m points and the Johnson scheme $J = J(7, 2)$, which is a coherent configuration of rank 3 on 21 points. Then the valences of \mathcal{X} are 1 and 10, and

$$n = 21m, \quad k = 10, \quad r = 3m.$$

It is easily seen that \mathcal{X} is a TI-scheme if and only if so is J . However, if J is a TI-scheme, then by Theorem 3.1, the group $\text{Aut}(J)$ contains a subgroup of order $21 \cdot 10$ acting on 21 points as a rank 3 group. But such a group must be solvable and primitive. Thus, the number 21 is a prime power, contradiction.

In fact, this example is a slight generalization of an example arising in studying pseudocyclic schemes [1, p.48]. As in that case, it is quite natural to introduce into consideration the indistinguishing number defined in (6).

Lemma 3.4. *In the above notation, let c be the indistinguishing number of the TI-scheme \mathcal{X} . Then $c \leq mk$ and this bound is tight.*

Proof. Denote by $\text{Fix}(x)$ the set of all points fixed by a permutation $x \in G$. Clearly, $\text{Fix}(1) = \Omega$. We claim that

$$(11) \quad |\text{Fix}(x)| \in \{0, m\}$$

for all non-identity permutations $x \in G$. Indeed, $\text{Fix}(x) = \emptyset$ if x belongs to no G_β , $\beta \in \Omega$. Suppose that the set $\text{Fix}(x)$ is not empty. Then $x \in G_\beta$ for some β . By the transitivity of G , without loss of generality, we may assume that $\beta = \alpha$. Then

$$\alpha S_1 \subseteq \text{Fix}(x),$$

see formula (10). Moreover, by Theorem 3.1, the group $H = G_\alpha$ acts semiregularly on the set $\Omega \setminus \alpha S_1$. This implies that $|\text{Fix}(x)| = |\alpha S_1| = m$, which proves claim (11).

On the other hand, in view of [12, Lemma 2.3], we have

$$(12) \quad c = \max_{g \in G \setminus H} \left| \bigcup_{x \in Hg} \text{Fix}(x) \right|.$$

By formula (11), this yields

$$c \leq \max_{g \in G \setminus H} \sum_{x \in Hg} |\text{Fix}(x)| \leq mk,$$

as required. Furthermore, the bound is attained if and only if there exists a coset $Hg \neq H$ such that

$$\text{Fix}(x) \neq \emptyset \quad \text{and} \quad \text{Fix}(x) \cap \text{Fix}(y) = \emptyset$$

for all distinct $x, y \in Hg$. Let us identify the points of Ω with the right H -cosets of G so that G acts on Ω by the right multiplications. Then the first condition holds if $Hzx = Hz$ for some $z \in G$, or, equivalently, if x^G intersects H . Next, the second condition is always true, whenever $|H| = 2$: indeed, if $Hx \in \text{Fix}(x) \cap \text{Fix}(y)$ for some $z \in G$, then

$$Hxz = Hz = Hzy,$$

which implies that the point stabilizer of Hx contains 1, x , and y , and hence $x = y$. Thus, the bound is attained if, for instance,

$$G = \text{Sym}(6), \quad H = \langle (1, 2)(3, 4) \rangle, \quad g = (1, 3)(2, 4).$$

In this case, the coset Hg consists of $x = g = (1, 3)(2, 4)$ and $y = (1, 4)(2, 3)$, and hence $x^z \in H$ for $z = (2, 3)$ and $y^z \in H$ for $z = (2, 4)$. \square

From Theorem 3.1 and Lemma 3.4, we immediately get the following statement, which, in some sense, justifies Definition 1.1.

Theorem 3.5. *Every TI-scheme is pseudo-TI.*

In general, the converse statement is not true, see Example 3.10 below. Theorem 1.2, which is proved later, shows that it becomes true if the index n/m is sufficiently large in comparison with k . The bound established in that theorem can be slightly improved if the pseudo-TI scheme in question is assumed to be schurian.

Proposition 3.6. *Every schurian pseudo-TI scheme of index greater than $2k(k-1)$ is TI.*

Proof. Let \mathcal{X} be a pseudo-TI scheme. Then k is the maximum valency of \mathcal{X} , $c \leq mk$, and, in view of the hypothesis, $n/m > 2k(k-1)$. It follows that

$$n > 2mk(k-1) \geq 2c(k-1).$$

Assume that \mathcal{X} is associated with a group G . Then according to [12, Theorem 3.1 and formula (2)], this implies that $H_\beta = G_{\alpha,\beta} = 1$ for some point β . Note that $\beta \notin \alpha S_1$, for otherwise $H = 1$. Consequently,

$$|H| = |\alpha s| = k,$$

where $s = r(\alpha, \beta)$. Since $S = S_1 \cup S_k$, the group H acts semiregularly on the set αS_k . Thus, H is a TI-subgroup of G by Theorem 3.1, as required. \square

3.2. Elementary coset schemes. One of the aims of this subsection is to determine the quasi-thin schemes, which are pseudo-TI. To this end, let $\mathcal{X} = (\Omega, S)$ be a homogeneous coherent configuration. Assume that there exists a set $T \subseteq S_1$ such that

$$(13) \quad ss^* = T \quad \text{for all } s \in S \setminus S_1$$

(see [14, Section 6.7]). In this case, we say that \mathcal{X} is an *elementary coset scheme*. Special classes of these schemes have been studied in [9, Section 3] and [4]. In both cases, \mathcal{X} was a wedge product of two regular schemes: one of them is the restriction of \mathcal{X} to αS_1 , and the other one is the quotient of \mathcal{X} modulo T .

Example 3.7. Let $C = \mathbb{Z}_n$ and $H \leq \text{Aut}(C)$ the group of prime order p dividing n . Denote by \mathcal{X} the TI-scheme associated with the group $G = C \rtimes H$ (see Example 3.2). Thus, $\Omega = C$ and S consists of the relations $s_c = (e, c)^G$, where e is the identity of C and $c \in C$; in particular,

$$S_1 = \{s_c : c \in C \text{ and } |c| \text{ divides } n/p\},$$

where $|c|$ is the order of c . It is straightforward to check that \mathcal{X} is an elementary coset scheme with $T = \{s_c : |c| = p\}$, see [4, Subsection 8.2].

Let \mathcal{X} be an elementary coset scheme. From [14, Lemma 6.7.1], it follows that $n_s = n_{ss^*}$ for all $s \in S$. Since $T \subseteq S_1$, this implies that $S = S_1 \cup S_k$, where $k = |T|$. Moreover, in view of (13), one can easily find that

$$c_{ss^*}^r = \begin{cases} k & \text{if } r \in S_1 \text{ and } s \in S_k, \\ 0 & \text{if } r \in S_k, \\ \delta_{r,1} & \text{if } r, s \in S_1, \end{cases}$$

where $\delta_{r,1}$ is the Kronecker symbol. It immediately follows that $c(r) = 0$ for all $r \in S \setminus S_1$ and $c(r) = n - m$ for $r \in (S_1)^\#$, where $m = |S_1|$. Thus, $c = n - m$. By the definition of pseudo-TI scheme, this proves the following statement.

Proposition 3.8. *An elementary coset scheme is pseudo-TI only if $n \leq m(k+1)$.*

Note that for the elementary coset scheme from Example 3.7, we have $k = p$ and $m = n/p$. In general, we do not know whether the inequality in Proposition 3.8 is sufficient for an elementary coset scheme to be pseudo-TI. However, the following statement shows that this is “almost” true for $k = 2$.

Theorem 3.9. *A schurian quasi-thin scheme is not TI if and only if it is elementary coset scheme and $n \geq 3m$ with possible exception $n = 3m$.³*

Proof. Let \mathcal{X} be a schurian quasi-thin scheme; in particular, $k = 2$ and

$$S = S_1 \cup S_2 \quad \text{and} \quad \text{Orb}(G_\alpha) = \{\alpha s : s \in S\},$$

where $G = \text{Aut}(\mathcal{X})$. The “if part” immediately follows from Proposition 3.8 and Theorem 3.5. To prove the “only if” part, assume that $n < 3m$ and verify that \mathcal{X} is a TI-scheme. Suppose first that \mathcal{X} has at least two orthogonals, i.e., the relations $s \in S^\#$ with $c(s) \neq 0$. Then by [11, Corollary 6.4], the coherent configuration \mathcal{X}_α is 1-regular. Therefore, the group G_α acts semiregularly on αS_2 . By Theorem 3.1, this implies that G_α is a TI-subgroup of G , and hence \mathcal{X} is a TI-scheme.

Now, we may assume that \mathcal{X} has exactly one orthogonal, say u . If $n_u = 2$, then u is the disjoint union of m cliques of size 3, and

$$\mathcal{X} \cong \mathcal{X}_1 \otimes \mathcal{X}_2,$$

where \mathcal{X}_1 and \mathcal{X}_2 are, respectively, trivial and regular coherent configurations of degrees 3 and m (see the proof of Theorem 5.2 in [11]). It follows that the group $\text{Aut}(\mathcal{X}) \cong \text{Aut}(\mathcal{X}_1) \times \text{Aut}(\mathcal{X}_2)$ is of order $6m = 2n$. Thus, $|G_\alpha| = |G|/n = 2$ and hence \mathcal{X} is a TI-scheme.

Finally, let $n_u = 1$. Then condition (13) is satisfied for $T = \{1, u\}$. Therefore \mathcal{X} is an elementary coset scheme of index 2 (the index can not be equal to 1, because $u \in S_1$). Then one can easily prove that

$$\mathcal{X}_\alpha \cong \mathcal{X}_1 \boxplus \mathcal{X}_2,$$

where \mathcal{X}_1 is the complete coherent configuration on αS_1 and \mathcal{X}_2 is a semiregular coherent configuration on αS_2 . This implies that \mathcal{X}_α is 1-regular and we are done by Theorem 3.1 as above. \square

Example 3.7 for $n = 3 \cdot 2^n$ and $p = 2$, gives a quasi-thin TI-scheme with $n = 3m$. On the other hand, the following example shows that there are quasi-thin pseudo-TI-schemes with $n = 3m$, which is not TI.

Example 3.10. Let G be a unique group of order $2^5 \cdot 3^2$ with elementary abelian socle of order 8 (in the GAP notation [5], this is the group [288, 859] with structure description “A4 x SL(2,3)”). It has the center of order 2 and exactly three non-normal subgroups of order 4 that lie in the socle and do not contain the center. Take one of them, say H ; the other two subgroups are conjugate. Then the action of G on the right H -cosets by the right multiplication is a transitive group of degree $n = 72$. The coherent configuration associated with G (in this action) is a quasi-thin elementary coset scheme \mathcal{X} with $n = 3m$. Therefore, it is pseudo-TI. On the other hand, $\text{Aut}(\mathcal{X}) = G$ and G contains no subgroup of index 2. This implies that \mathcal{X} can not be a TI-scheme.

As it was shown in [11], any non-schurian quasi-thin scheme \mathcal{X} has 2 or 3 orthogonals, they generate the Klein group, and the index of \mathcal{X} equals 4 or 7. It is very likely that such a scheme is pseudo-TI.

³As we will see below, for $n = 3m$, there exist TI and non-TI quasi-thin elementary coset schemes.

3.3. Further examples of pseudo-TI p -schemes. In general, it is not easy to find or estimate the indistinguishing number of a coherent configuration \mathcal{X} and hence to check whether or not \mathcal{X} is a pseudo-TI scheme. However, if $mk \geq n$, then, obviously, inequality (2) holds. Therefore, \mathcal{X} is pseudo-TI if and only if $S = S_1 \cup S_k$. Such schemes do exist as the following direct interpretation of a construction in [8] shows.

Theorem 3.11. *For every prime $p \equiv 3 \pmod{4}$ and each of two non-abelian group H of order p^3 , there exists a non-schurian pseudo-TI scheme with*

$$(n, m, k) = (p^3, p^2, p)$$

admitting H as a regular automorphism group.

Another example comes from the Suzuki group $G = \text{Sz}(q)$, where $q = 2^{2n+1}$. The following facts are well known, see, e.g., [2]. The group G is of order $q^2(q^2+1)(q-1)$, has q^2+1 Sylow 2-subgroups, and any two distinct Sylow 2-subgroups intersect at identity. In particular, each $H \in \text{Syl}_2(G)$ is a TI-subgroup of G . In addition, $|N_G(H) : H| = q-1$. Thus, the coherent configuration associated with the action of G on the right H -cosets by the right multiplication is a TI-scheme with

$$(n, m, k) = (q^2(q-1), q-1, q^2).$$

It should be mentioned that H is a non-abelian 2-group. Some other examples of simple groups with TI-subgroups can be found in [13].

4. MATCHING CONFIGURATIONS

Let $\mathcal{M} = (\Delta, M)$ be a partial coherent configuration, the fibers Δ_x of which are indexed by the elements of a set X . Denote by $M(x, y)$ the set of all its basis relations contained in $\Delta_x \times \Delta_y$. In what follows, for all $x, y \in X$, we set $1_x = 1_{\Delta_x}$ and write $x \sim y$ if $M(x, y)$ is a partition of $\Delta_x \times \Delta_y$ into matchings.

Definition 4.1. *We say that \mathcal{M} is a matching configuration if for every $x, y \in X$ with $M(x, y) \neq \emptyset$, either $x \sim y$ or $x = y$ and $M(x, y) = \{1_x\}$.*

Any semiregular coherent configuration is, obviously, a matching configuration. In general, with a matching configuration \mathcal{M} , we associate an undirected graph $D = D(\mathcal{M})$ with vertex set X , in which the vertices x and y are adjacent if and only if $x \sim y$. In particular, we do not exclude loops. The following two statements describe the adjacency and triangles in D :

$$(14) \quad x \sim y \quad \Leftrightarrow \quad |\Delta_x| = |M(x, y)| = |M(y, x)| = |\Delta_y|,$$

$$(15) \quad x \sim y \sim z \sim x \quad \Rightarrow \quad M(x, y) \cdot M(y, z) = M(x, z).$$

The first of them is obvious, whereas the second one follows from condition (C3) in the definition of partial coherent configuration.

The matching configuration \mathcal{M} is said to be *saturated* if for any set $Y \subseteq X$ with at most 4 elements, there exists a vertex of D adjacent with every vertex of Y . Note that in this case, any two vertices $x, y \in X$ are connected by a 2-path $P = (x, z, y)$: take $Y = \{x, y\}$. In particular, D is a graph of diameter at most 2.

Theorem 4.2. *Let $\mathcal{M} = (\Delta, M)$ be a saturated matching configuration, and let $\mathcal{Y} = (\Delta, T)$ be the coherent closure of \mathcal{M} . Then $T = (M \cdot M)^\#$.*

Proof. We recall that the fibers $\Delta_x \in \Phi(\mathcal{M})$ are indexed by the elements $x \in X$. Since \mathcal{M} is saturated, formula (14) implies that the number $|\Delta_x|$ does not depend on $x \in X$; denote this number by k . For any d -path $P = (x_1, \dots, x_{d+1})$ of the graph D , set

$$M_P = M(x_1, x_2) \cdot \dots \cdot M(x_d, x_{d+1}).$$

Clearly, M_P consists of matchings contained in $\Delta_{x_1} \times \Delta_{x_{d+1}}$.

Lemma 4.3. *For every path P , the set M_P is a partition with k classes. Moreover, this partition does not depend on the choice of the path P connecting x_1 and x_{d+1} .*

Proof. To prove the first statement, it suffices to verify that the equalities

$$(16) \quad a \cdot M(x_2, x_3) = M(x_1, x_2) \cdot M(x_2, x_3) = M(x_1, x_2) \cdot b$$

hold for any $a \in M(x_1, x_2)$ and $b \in M(x_2, x_3)$. Note that if the first equality is true for all 2-paths $P = (x_1, x_2, x_3)$ and a , then applying it to the path $P^* = (x_3, x_2, x_1)$, we have

$$M(x_1, x_2) \cdot b = (b^* \cdot M(x_2, x_1))^* = (M(x_3, x_2) \cdot M(x_2, x_1))^* = M(x_1, x_2) \cdot M(x_2, x_3).$$

Thus, it suffices to verify the first equality in (16), or equivalently, that for every $a' \in M(x_1, x_2)$ and $b' \in M(x_2, x_3)$, there exists $b \in M(x_2, x_3)$ such that

$$(17) \quad a \cdot b = a' \cdot b'.$$

To do this, we observe that by the saturation property of \mathcal{M} , there exists a vertex $y \in X$ adjacent to each of the vertices x_1, x_2, x_3 in the graph D . It follows that $x_1 \sim y \sim x_2 \sim x_1$ and $x_2 \sim y \sim x_3 \sim x_2$. By formula (15), this implies that

$$(18) \quad M(x_1, y) \cdot M(y, x_2) = M(x_1, x_2), \quad M(x_2, y) \cdot M(y, x_3) = M(x_2, x_3).$$

Using these equalities, we successively find $u \in M(x_1, y)$ and $t \in M(x_2, y)$ such that $a' \cdot t = u$, and then $v \in M(y, x_3)$ such that $t \cdot v = b'$. Then

$$(19) \quad a' \cdot b' = (u \cdot t^*) \cdot (t \cdot v) = u \cdot v.$$

Using equalities (18) again, we first find $s \in M(x_2, y)$ such that $a \cdot s = u$, and then $b \in M(x_2, x_3)$ such that $s^* \cdot b = v$ (the obtained configuration is depicted at Fig. 1). Thus, from (19), it follows that

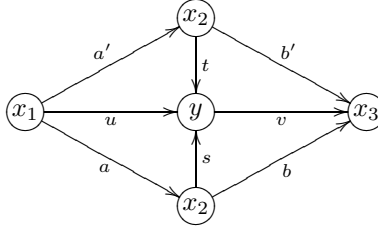


FIGURE 1.

$$a' \cdot b' = u \cdot v = (a \cdot s) \cdot (s^* \cdot b) = a \cdot b,$$

which proves (17). This completes the proof of (16), and, hence, the first statement.

To prove the second statement, let P and P' be a d - and a d' -path connecting the elements $u = x_1 = x'_1$ and $v = x_{d+1} = x'_{d'+1}$. Without loss of generality,

we can assume that $d + d' \geq 3$. By formula (15), the required statement follows for $d + d' = 3$. Suppose that $d + d' \geq 4$. Then since the matching configuration \mathcal{M} is saturated, there exists a vertex $w \in X$ adjacent with each of the vertices $\{u, x_2, x_3, x'_2\}$ in the graph D . By formula (16) applied to (u, w, x_2) and (x_2, w, x_3) , we have

$$\begin{aligned} M_P &= M(u, x_2) \cdot M(x_2, x_3) \cdot M_{Q_1} = (M(u, w) \cdot a) \cdot (a^* \cdot M(w, x_3)) \cdot M_{Q_1} = \\ &= M(u, w) \cdot M(w, x_3) \cdot M_{Q_1} = M(u, w) \cdot M_Q, \end{aligned}$$

where $Q_1 = (x_3, \dots, v)$, $Q = (w, x_3, \dots, v)$, and $a \in M(w, x_2)$. Similarly, one can prove that $M_{P'} = M(u, w) \cdot M_{Q'}$, where $Q' = (w, x'_2, \dots, v)$. Note that Q and Q' are respectively, $(d - 1)$ - and d' -paths connecting w and v . By the induction hypothesis, this implies that $M_Q = M_{Q'}$ and hence

$$M_P = M(u, w) \cdot M_Q = M(u, w) \cdot M_{Q'} = M_{P'}$$

as required. \square

In what follows, we keep the notation of Theorem 4.2. Let us represent the set $T' = (M \cdot M)^\#$ as the union of the sets

$$T'(x, y) = \bigcup_{z \in N(x, y)} M(x, z) \cdot M(z, y), \quad x, y \in X,$$

where $N(x, y) = \{z \in X : x \sim z \sim y\}$. Note that the latter set is not empty for all x and y , because the matching configuration \mathcal{M} is saturated. Moreover, if $z \in N(x, y)$, then $M(x, z) \cdot M(z, y) = M_P$, where $P = (x, z, y)$ is a 2-path. By Lemma 4.3, this implies that $T'(x, y)$ is the partition of $\Delta_x \times \Delta_y$ into matchings. In particular,

$$(20) \quad 1_x = u \cdot u^* \in M(x, z) \cdot M(z, x) = T'(x, x),$$

where $u \in M(x, z)$.

Lemma 4.4. *The pair $\mathcal{Y}' = (\Delta, T')$ is a coherent configuration.*

Proof. By the above remarks, the set T' is a partition of $\Delta \times \Delta$ into matchings and $1_\Delta \in (T')^\cup$. It is easily seen that $(T')^* = T'$. Thus, it suffices to verify that if $u, v \in T'$ and $u \cdot v \neq \emptyset$, then $u \cdot v \in T'$. To this end, let $u \in M_P$ and $v \in M_Q$, where P and Q are 2-paths of the graph D . Since $u \cdot v \neq \emptyset$, the last vertex of P coincides with the first vertex of Q . By Lemma 4.3, this implies that

$$M_P \cdot M_Q = M_{P \cdot Q},$$

where $P \cdot Q$ is the 4-path of D consisting of the vertices of P followed by the vertices of Q (the last vertex of P is identified with the first vertex of Q). Thus, $u \cdot v \in M_{P \cdot Q}$ belongs to T' , as required. \square

To complete the proof of Theorem 4.2, we note that the coherent configuration \mathcal{Y}' defined in Lemma 4.4 contains the coherent closure \mathcal{Y} of M . Indeed, let $x \sim y$. Then

$$M(x, y) = 1_x \cdot M(x, y) \subset T'.$$

This implies that $M \subseteq T' \subseteq (T')^\cup$. Thus, the claim follows, because \mathcal{Y} is the smallest coherent configuration, for which every relation of M is the union of basis relations. Conversely, as is easily seen, every relation in the set $M \cdot M$ is contained in T^\cup . By the definition of T' , this implies that $T' \subseteq T^\cup$, i.e., $\mathcal{Y}' \leq \mathcal{Y}$. Thus, $\mathcal{Y}' = \mathcal{Y}$, as required. \square

The set T in Theorem 4.2 consists of matchings. Therefore, $n_t = 1$ for all $t \in T$. By formula (20), this proves the following statement.

Corollary 4.5. *In the notation of Theorem 4.2, the coherent configuration \mathcal{Y} is semiregular and $\Phi(\mathcal{Y}) = \Phi(\mathcal{M})$.*

From Corollary 4.5 and Theorem 2.3, it follows that every saturated matching configuration as well as its coherent closure are schurian. Since the latter one is also separable, the following statement says that, in fact, every saturated matching configuration is also separable (cf. Lemma 2.2).

Theorem 4.6. *Let $\mathcal{M} = (\Delta, M)$ and $\mathcal{M}' = (\Delta', M')$ be saturated matching configurations with coherent closures $\mathcal{Y} = (\Delta, T)$ and $\mathcal{Y}' = (\Delta', T')$, respectively. Let $\varphi : M \rightarrow M'$, $u \mapsto u'$ be an algebraic isomorphism from \mathcal{M} onto \mathcal{M}' . Then*

$$T = (M \cdot M)^\# \quad \text{and} \quad T' = (M' \cdot M')^\#,$$

and the mapping

$$(21) \quad \psi : T \rightarrow T', \quad u \cdot v \mapsto u' \cdot v'$$

is a well-defined bijection. Moreover, $\psi|_M = \varphi$ and ψ is an algebraic isomorphism from \mathcal{Y} to \mathcal{Y}' .

Proof. The first statement immediately follows from Theorem 4.2. To verify that the mapping ψ is well-defined, suppose that $b_1 \cdot c_1 = b_2 \cdot c_2$ for some $b_1, c_1, b_2, c_2 \in M$. Let $x, y, z_1, z_2 \in X$ be such that

$$b_i \in M(x, z_i) \quad \text{and} \quad c_i \in M(z_i, y), \quad i = 1, 2.$$

In particular, $x \sim z_i \sim y$ for each i . Since the matching configuration \mathcal{M} is saturated, there exists a vertex $z \in X$ adjacent to each of the vertices x, z_1, z_2, y in the graph D . By formula (15), this implies that

$$(22) \quad M(x, z_i) = M(x, z) \cdot M(z, z_i) \quad \text{and} \quad M(z, y) = M(z, z_i) \cdot M(z_i, y)$$

for each i . Take any $a_1 \in M(x, z)$. Then the first equality in (22) implies that

$$(23) \quad d_1 := a_1^* \cdot b_1 \in M(z, z_1) \quad \text{and} \quad d_2 := a_1^* \cdot b_2 \in M(z, z_2),$$

whereas by the second equality in (22), we have

$$(24) \quad a_2 := d_1 \cdot c_1 \in M(z, y).$$

Thus,

$$(25) \quad a_1 \cdot a_2 = (a_1 \cdot d_1) \cdot (d_1^* \cdot a_2) = b_1 \cdot c_1 = b_2 \cdot c_2 = a_1 \cdot d_2 \cdot c_2,$$

whence $c_2 = d_2^* \cdot a_2$ (see Fig. 2).

On the other hand, the algebraic isomorphism φ induces a bijection from $\Phi(\mathcal{M})$ onto $\Phi(\mathcal{M}')$ and hence a bijection $x \mapsto x'$ from X onto X' . In these notation,

$$M(x, y)^\varphi = M'(x', y').$$

In particular, $x \sim y$ if and only if $x' \sim y'$ for all $x, y \in X$. Thus, the bijection $x \mapsto x'$ is a graph isomorphism from $D(\mathcal{M})$ onto $D(\mathcal{M}')$. Moreover, equality (15) implies that if $x \sim y \sim z \sim x$ and $a \in M(x, y)$, $b \in M(y, z)$, then $(a \cdot b)' = a' \cdot b'$. Thus, in view of formulas (23), (24), and (25), we have

$$b'_1 \cdot c'_1 = (a_1 \cdot d_1)' \cdot (d_1^* \cdot a_2)' = a'_1 \cdot (d'_1 \cdot (d'_1)^*) \cdot a'_2 = a'_1 \cdot a'_2 =$$

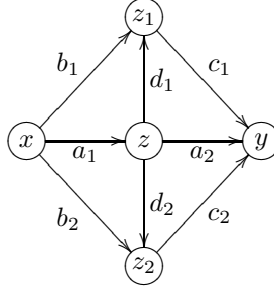


FIGURE 2.

$$(b_2 \cdot d_2^*)' \cdot (d_2 \cdot c_2)' = b_2' \cdot (d_2' \cdot (d_2^*)') \cdot c_2' = b_2' \cdot c_2',$$

which proves that ψ is a well-defined bijection. Note that if $a \in M(x, y)$ for some x and y , then

$$a^\psi = (1_x \cdot a)^\psi = 1_{x'} \cdot a' = a' = a^\varphi.$$

Thus, $\psi|_M = \varphi$.

To prove that ψ is an algebraic isomorphism from \mathcal{Y} onto \mathcal{Y}' , we note that if $a, b \in T$, then either $a \cdot b \in T$ or $a \cdot b = \emptyset$. Thus, it suffices to verify that

$$(26) \quad (a \cdot b)^\psi = a^\psi \cdot b^\psi, \quad a, b \in T, \quad a \cdot b \neq \emptyset.$$

To this end, take such relations a and b . Then

$$a \in M(x, z_1) \cdot M(z_1, z') \quad \text{and} \quad b \in M(z', z_2) \cdot M(z_2, y)$$

for appropriate $x, z_1, z', z_2, y \in X$. Since \mathcal{M} is saturated, we may assume that $z_1 = z_2$; denote this element by z . Then $x \sim z' \sim z \sim x$ and $y \sim z' \sim z \sim y$. In view of formula (15), one can find $c_1 \in M(x, z)$, $c_2 \in M(z, z')$, and $c_3 \in M(z, y)$ such that

$$c_1 \cdot c_2 = a \quad \text{and} \quad c_2 \cdot c_3 = b$$

(see Fig. 3). Besides, by the definition of ψ , we have $(c_1 \cdot c_3)^\psi = c_1^\varphi \cdot c_3^\varphi$. Since

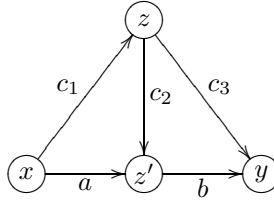


FIGURE 3.

$\psi|_M = \varphi$, we obtain

$$(a \cdot b)^\psi = (c_1 \cdot c_2 \cdot c_2^* \cdot c_3)^\psi = (c_1 \cdot c_3)^\psi = c_1^\varphi \cdot c_3^\varphi = c_1^\psi \cdot c_3^\psi =$$

$$c_1^\psi \cdot c_2^\psi \cdot (c_2^*)^\psi \cdot c_3^\psi = (c_1 \cdot c_2)^\psi \cdot (c_2^* \cdot c_3)^\psi = a^\psi \cdot b^\psi,$$

which completes the proof of (26). \square

5. MATCHING SUBCONFIGURATION OF A COHERENT CONFIGURATION

5.1. Main statement. Let $\mathcal{X} = (\Omega, S)$ be a homogeneous coherent configuration with maximum valency k . Define a binary relation \sim on the set $X = S_k$ by setting

$$(27) \quad x \sim y \quad \Leftrightarrow \quad |x^*y| = k.$$

Formally, this relation is different from that introduced in Section 4, but as we will see, they are similar in some sense. The relation (27) is symmetric, but, in general, not reflexive or transitive. Formula (5) implies that

$$k^2 = n_{x^*}n_y = \sum_{z \in x^*y} n_z c_{x^*y}^z = \sum_{z \in x^*y} n_{y^*} c_{z^*x^*}^y = k \sum_{z \in x^*y} c_{xz}^y$$

(here we also use that $n_x = n_{x^*} = n_y = n_{y^*} = k$). Since $c_{xz}^y \geq 1$ for all $z \in x^*y$, we conclude that

$$(28) \quad x \sim y \quad \Leftrightarrow \quad c_{xz}^y = 1 \text{ for all } z \in x^*y.$$

Fix a point $\alpha \in \Omega$. Then this formula implies also that $x \sim y$ if and only if each nonempty relation $r_{x,y}$ defined in Lemma 2.1 is a matching. Set

$$(29) \quad \Delta_\alpha = \alpha X \quad \text{and} \quad M_\alpha = \bigcup_{x,y \in X} M_\alpha(x,y),$$

where

$$(30) \quad M_\alpha(x,y) = \begin{cases} S_\alpha(x,y) & \text{if } x \sim y, \\ \emptyset & \text{if } x \not\sim y \text{ and } x \neq y, \\ \{1_{\alpha x}\} & \text{if } x \not\sim y \text{ and } x = y, \end{cases}$$

with $S_\alpha(x,y) = \{r_{x,y} : r \in x^*y\}$. Thus, the pair $\mathcal{M}_\alpha = (\Delta_\alpha, M_\alpha)$ is close to be a matching configuration, see Definition 4.1. The problem is that, in general, \mathcal{M}_α is not necessarily a partial coherent configuration: formula (15) is not always true.

Theorem 5.1. *Let $\mathcal{X} = (\Omega, S)$ be a homogeneous coherent configuration with maximum valency $k > 1$ and indistinguishing number c . Suppose that*

$$(31) \quad |X| > 6c(k-1),$$

where $X = S_k$. Then given a point $\alpha \in \Omega$, the pair \mathcal{M}_α is a saturated matching configuration; in particular, $\Phi(\mathcal{M}_\alpha) = \{\alpha x : x \in X\}$. Moreover,

$$(32) \quad S_\alpha(x,y) \subseteq (M_\alpha(x,z) \cdot M_\alpha(z,y))^\cup$$

for all $x, y, z \in X$ such that $x \sim z \sim y$.

We prove Theorem 5.1 in Subsection 5.4. The key points of the proof are the lemmas proved in Subsections 5.2 and 5.3.

5.2. Neighborhoods. In this subsection, we are interested in estimating the cardinality of the *neighborhood* $N(Y)$ of a set $Y \subseteq X$ with respect to the relation \sim that is defined as follows:

$$(33) \quad N(Y) = \{x \in X : x \sim y \text{ for all } y \in Y\}.$$

In the sequel, we set $N(x, y, \dots) = N(\{x, y, \dots\})$. The following statement gives a lower bound for the cardinality of the neighborhood.

Lemma 5.2. *If $Y \subseteq X$, then $|N(Y)| \geq |X| - c(k-1)|Y|$.*

Proof. Obviously,

$$|N(Y)| \geq |X| - |Y| \max_{y \in Y} |X_y|,$$

where X_y is the set of all $x \in X$ such that $y \not\sim x$. Therefore, it suffices to verify that for any $y \in X$,

$$(34) \quad |X_y| \leq c(k-1).$$

To do this fix a relation $y \in X$, a point $\alpha \in \Omega$, and denote by Λ_y the set of all pairs of distinct points of αy . From (28), it follows that $x \in X_y$ only if $c_{yz}^x > 1$ for some $z \in y^*x$. In the latter case, for each $\beta \in \alpha x$, there exists a pair $(\gamma, \delta) \in \Lambda_y$ such that

$$(35) \quad r(\gamma, \beta) = z = r(\delta, \beta).$$

It follows that the set $T_{x,y}$ of all such triples (β, γ, δ) contains at least $|\alpha x| = n_x = k$ elements. Therefore, the union of all sets $T_{x,y}$ with $x \in X_y$ contains at least $k|X_y|$ elements. On the other hand,

$$\bigcup_{x \in X_y} T_{x,y} = \bigcup_{e \in \Lambda_y} T_e,$$

where T_e is the set of all triples belonging to the left-hand side and the second and third entries of which forms the pair e . Moreover, the number of nonempty summands on the right-hand side is at most $|\Lambda_y| = k(k-1)$. Thus, there exists a pair $e \in \Lambda_y$ such that

$$|T_e| \geq \frac{1}{k(k-1)} \sum_{x \in N_y} |T_{x,y}| \geq \frac{k|X_y|}{k(k-1)}.$$

Note that in view of (35) if $e = (\gamma, \delta)$, then T_e is contained in the set $\Omega_{\gamma,\delta}$ defined in formula (6). Consequently,

$$c \geq |\Omega_{\gamma,\delta}| \geq |T_e| \geq \frac{|X_y|}{k-1}.$$

which proves formula (34). \square

5.3. Special elements. Let $x, y, z \in X$ and $r, s, t \in S$ be such that

$$(36) \quad x \sim z \sim y \quad \text{and} \quad r \in x^*z, \quad s \in z^*y, \quad t \in x^*y.$$

Then the relations $r_{x,z} \in S_\alpha(x, z)$ and $s_{z,y} \in S_\alpha(z, y)$ are matchings, whereas the relation $t_{x,y} \in S_\alpha(x, y)$ is not necessarily a matching.

Definition 5.3. An element $q \in N(x, y, z)$ is said to be *special with respect to the 6-tuple $T = (x, r, z, s, y, t)$* , or *T -special* if there exist elements $u \in x^*q$, $v \in z^*q$, and $w \in y^*q$ such that

$$(37) \quad uv^* \cap x^*z = \{r\} \quad \text{and} \quad vw^* \cap z^*y = \{s\} \quad \text{and} \quad uw^* \cap x^*y = \{t\},$$

see the configuration depicted in Fig.4.

It follows from the definition that $q \sim x$, $q \sim y$, and $q \sim z$. The following two statements reveal the property of special elements, to be used in the sequel, and provide a sufficient condition for their existence.

Lemma 5.4. In the above notation, assume that the set $N(x, y, z)$ contains a T -special element. Then $r_{x,z} \cdot s_{z,y} \subseteq t_{x,y}$.

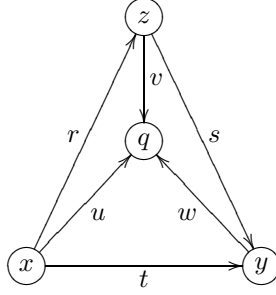


FIGURE 4.

Proof. Let q be a T -special element. Then there exist elements $u \in x^*q$, $v \in z^*q$, and $w \in y^*q$, for which relations (37) hold. It follows that

$$(38) \quad u_{x,q} \cdot v_{q,z}^* \subseteq r_{x,z}, \quad v_{z,q}^* \cdot w_{q,y}^* \subseteq s_{z,y}, \quad u_{x,q} \cdot w_{q,y}^* \subseteq t_{x,y}.$$

On the other hand, the relations $u_{x,q} \cdot v_{q,z}^*$ and $r_{x,z}$ are matchings, because $x \sim q \sim z$, and $x \sim z$. Therefore, by the first inclusion in (38), we conclude that $u_{x,q} \cdot v_{q,z}^* = r_{x,z}$. Similarly, $v_{z,q}^* \cdot w_{q,y}^* = s_{z,y}$. Thus, by the third inclusion in (38), we have

$$r_{x,z} \cdot s_{z,y} = (u_{x,q} \cdot v_{q,z}^*) \cdot (v_{z,q}^* \cdot w_{q,y}^*) = u_{x,q} \cdot w_{q,y}^* \subseteq t_{x,y},$$

which proves the required inclusion. \square

Lemma 5.5. Let $T = (x, r, z, s, y, t)$, where $x, y, z \in X$ and $r, s, t \in S$ are such that formula (36) holds. Assume that

$$(39) \quad |N(x, y, z)| > 3c(k-1) \quad \text{and} \quad (r_{x,z} \cdot s_{z,y}) \cap t_{x,y} \neq \emptyset.$$

Then the set $N(x, y, z)$ contains a T -special element.

Proof. By the right-hand side relation in (39), there exist points $\beta \in \alpha x$, $\gamma \in \alpha z$, and $\delta \in \alpha y$ such that

$$(40) \quad (\beta, \gamma) \in r, \quad (\gamma, \delta) \in s, \quad (\beta, \delta) \in t.$$

For every relation $q \in N(x, y, z)$ and every point $\mu \in \alpha q$ set

$$(41) \quad u(q, \mu) := r(\beta, \mu), \quad v(q, \mu) := r(\gamma, \mu), \quad w(q, \mu) := r(\delta, \mu).$$

Then, obviously,

$$(42) \quad r \in uv^* \cap x^*z, \quad s \in vw^* \cap z^*y, \quad t \in uw^* \cap x^*y.$$

where $u = u(q, \mu)$, $v = v(q, \mu)$, and $w = w(q, \mu)$. Therefore, if q is not T -special, then for each $\mu \in \alpha q$, there exists a basis relation $a = a(q, \mu)$ such that

$$(43) \quad a \in (uv^* \cap x^*z) \setminus \{r\} \quad \text{or} \quad a \in (vw^* \cap z^*y) \setminus \{s\} \quad \text{or} \quad a \in (uw^* \cap x^*y) \setminus \{t\}.$$

Note that each of the sets x^*z , z^*y , x^*y consists of at most k relations. So the relation $a(q, \mu)$ is one of the $3(k-1)$ relations contained in the set $x^*z \cup z^*y \cup x^*y$, which does not depend on q and μ . Thus, if S_T is the set of all non- T -special relations q and P_a is the set of all pairs $(q, \mu) \in S_T \times \alpha q$ with $a = a(q, \mu)$, then there exists $a \in x^*z \cup z^*y \cup x^*y$ such that

$$(44) \quad |P_a| \geq \frac{k|S_T|}{3(k-1)}.$$

Without loss of generality, we may assume that $a \in x^*z$. Let now $(q, \mu) \in P_a$. Then by the definition of $u = u(q, \mu)$ and $v = v(q, \mu)$, we have $\mu \in \beta u \cap \gamma v$. Since also $a \in uv^*$ (see (43)), there exists a point $\nu(q, \mu)$ belonging to the set $\mu u^* \cap \gamma a^*$, see the configuration depicted in Fig. 5. Note that every point $\nu(q, \mu)$ belongs to

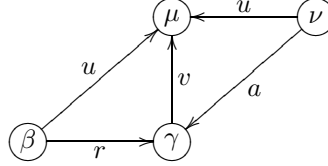


FIGURE 5.

the set γa^* of cardinality at most k . Therefore, there exists a point $\nu \in \gamma a$ such that the set $P_{a,\nu}$ of all pairs $(q, \mu) \in P_a$ with $\nu = \nu(q, \mu)$ contains at least $|P_a|/k$ elements. Taking into account that μ is contained in the set $\Omega_{\beta,\nu}$ defined by (6), we conclude by (44) that

$$c \geq |\Omega_{\beta,\nu}| = |P_{a,\nu}| \geq \frac{|P_a|}{k} \geq \frac{k|S_T|}{3k(k-1)},$$

whence $|S_T| \leq 3c(k-1)$. By the hypothesis of the lemma, this implies that the set $N(x, y, z) \setminus S_T$ is not empty. Since every element of this set is T -special, we are done. \square

5.4. Proof of Theorem 5.1. Let $x, y, z \in X$ be such that $x \sim y \sim z$. Then given matchings $a \in M(x, z)$ and $b \in M(z, y)$, there exists a relation $c \in S(x, y)$ such that $a \cdot b$ intersects c . It follows that

$$(r_{x,z} \cdot s_{z,y}) \cap t_{x,y} \neq \emptyset,$$

where the relations $r, s, t \in S$ are defined by the conditions $a = r_{x,z}$, $b = s_{z,y}$, and $c = t_{x,y}$, respectively. On the other hand by the theorem hypothesis and Lemma 5.2,

$$|N(x, y, z)| \geq |X| - 3c(k-1) > 3c(k-1).$$

Thus, the hypothesis of Lemma 5.5 holds for $T = (x, r, z, s, y, t)$. This shows that the set $N(x, y, z)$ contains a T -special element. By Lemma 5.4, this implies that

$$a \cdot b = r_{x,z} \cdot s_{z,y} \subseteq t_{x,y}.$$

This proves formula (32). Moreover, if $x \sim z$, then $t_{x,y}$ is a matching of size k . Since $a \cdot b$ is also a matching of the same size, we conclude that $a \cdot b = c$. Thus, condition (15) is satisfied and hence \mathcal{M}_α is a matching configuration. To prove that it is saturated, it suffices to verify that the set $N(Y)$ is not empty for all $Y \subseteq X$ with $|Y| \leq 4$. But in this case,

$$|N(Y)| \geq |X| - 4c(k-1) > 2c(k-1) > 0$$

by formula (31) and Lemma 5.2. \square

5.5. Algebraic isomorphisms. In this subsection, we fix an algebraic isomorphism $\varphi : \mathcal{X} \rightarrow \mathcal{X}'$, $r \mapsto r'$, where \mathcal{X} is a coherent configuration satisfying condition (31) and $\mathcal{X}' = (\Omega', S')$ is an arbitrary coherent configuration.

Theorem 5.6. *In the above notation and assumptions, the coherent configuration \mathcal{X}' satisfies condition (31). Moreover, given points $\alpha \in \Omega$ and $\alpha' \in \Omega'$, the mapping*

$$\varphi_{\alpha, \alpha'} : M_\alpha \rightarrow M'_{\alpha'}, \quad r_{x, y} \mapsto r'_{x', y'}$$

is an algebraic isomorphism from \mathcal{M}_α to $\mathcal{M}'_{\alpha'}$. Moreover, if $a \in M_\alpha$ and $a \subset r \in S$, then $a^{\varphi_{\alpha, \alpha'}} \subset r^{\varphi}$.

Proof. Clearly, the rank, maximal valency, and indistinguishing number of \mathcal{X}' are the same as of \mathcal{X} . By Theorem 5.1, given $\alpha' \in \Omega'$, one can define a saturated matching configuration $\mathcal{M}'_{\alpha'} = (\Delta'_{\alpha'}, M'_{\alpha'})$ exactly in same way as for \mathcal{X} . Since the algebraic isomorphism φ takes $X := S_k$ to $X' := S'_k$ and $r_{x, y}$ is not empty if and only if so is $r'_{x', y'}$, the mapping $\psi = \varphi_{\alpha, \alpha'}$ is a bijection. We need the following lemma.

Lemma 5.7. *Let $x, y, z \in X$ be such that $x \sim y \sim z$, and let $r \in x^*z$, $s \in z^*y$. Then $r_{x, z} \cdot s_{z, y} \subseteq t_{x, y}$ for some $t \in x^*y$. Moreover*

$$(r_{x, z})^\psi \cdot (s_{z, y})^\psi \subseteq t'_{x', y'}.$$

Proof. Let $a = r_{x, z}$, $b = s_{z, y}$, and $c = a \cdot b$. Then $c \in M_\alpha(x, z) \cdot M_\alpha(z, y)$. By formula (32), this implies that $c \subseteq t_{x, y}$, where t is a unique relation in x^*y that intersects c . This proves the first statement. To prove the second one, we note that

$$r_{x, z} \cdot s_{z, y} \cap t_{x, y} \neq \emptyset \quad \text{and} \quad |N(x, y, z)| > 3c(k-1),$$

where the latter follows from Lemma 5.2. In view of Lemma 5.5, this implies that $N(x, y, z)$ contains a T -special element q , where $T = (x, r, z, s, y, t)$. This means that formula (37) holds and hence

$$u'(v')^* \cap (x')^* z' = \{r'\} \quad \text{and} \quad v'(w')^* \cap (z')^* y' = \{s'\} \quad \text{and} \quad u'(w')^* \cap (x')^* y' = \{t'\}$$

for suitable relations $u' \in (x')^* q'$, $v' \in (z')^* q'$, and $w' \in (y')^* q'$. Therefore, q' is a T' -special element, where $T' = (x', r', z', s', y', t')$. By Lemma 5.4, this shows that

$$r'_{x', z'} \cdot s'_{z', y'} \subseteq t'_{x', y'},$$

as required. \square

To complete the proof of Theorem 5.6, assume that in Lemma 5.7, $x \sim y$. Then $r_{x, z} \cdot s_{z, y} = t_{x, y}$ is a matching. Since $(t_{x, y})' = t'_{x', y'}$ by the definition of $\varphi_{\alpha, \alpha'}$, we obtain

$$(r_{x, z})^\psi \cdot (s_{z, y})^\psi = (r_{x, z} \cdot s_{z, y})^\psi.$$

Thus, ψ is an algebraic isomorphism from \mathcal{M}_α onto $\mathcal{M}'_{\alpha'}$, as required. \square

6. PROOF OF THEOREM 1.3

By the theorem hypothesis, the set $X := S_k$ coincides with $S \setminus S_1$. Since also $|S|$ is greater than $m + 6c(k-1)$, we have

$$(45) \quad |X| = |S_k| = |S| - |S_1| > m + 6c(k-1) - m = 6c(k-1),$$

i.e., inequality (31) holds. By Theorem 5.1, this implies that for each point $\alpha \in \Omega$, the pair $\mathcal{M}_\alpha = (\Delta_\alpha, M_\alpha)$ defined by (29) is a saturated matching configuration with fibers $\Omega_x = \alpha x$, $x \in X$, and formula (32) holds.

Lemma 6.1. *Let $\mathcal{Y}_\alpha = (\Delta_\alpha, T_\alpha)$ be the coherent closure of M_α , and let \mathcal{D}_α be the complete coherent configuration on $\Omega \setminus \Delta_\alpha$. Then*

$$(46) \quad \mathcal{X}_\alpha = \mathcal{D}_\alpha \boxplus \mathcal{Y}_\alpha.$$

In particular, $\Phi(\mathcal{X}_\alpha) = \{\alpha s : s \in S\}$.

Proof. By the definition of M_α , we have $M_\alpha \subseteq (S_\alpha)^\cup$, where S_α is the set of basis relations of the coherent configuration \mathcal{X}_α . By formula (32) and Theorem 4.2, this implies that

$$T_\alpha = (M_\alpha \cdot M_\alpha)^\# \subset M_\alpha \cdot M_\alpha \subseteq (S_\alpha)^\cup.$$

Thus, $(\mathcal{X}_\alpha)_{\Delta_\alpha} \geq \mathcal{Y}_\alpha$ by the minimality of the coherent configuration \mathcal{Y}_α . Moreover, $\Omega \setminus \Delta_\alpha = \alpha S_1$ and $\alpha s \in \Phi(\mathcal{X}_\alpha)$ for all $s \in S_1$. Therefore

$$\mathcal{X}_\alpha = (\mathcal{X}_\alpha)_{\Omega \setminus \Delta_\alpha} \boxplus (\mathcal{X}_\alpha)_{\Delta_\alpha} \geq \mathcal{D}_\alpha \boxplus \mathcal{Y}_\alpha \geq \mathcal{X},$$

and we are done by the minimality of the coherent configuration \mathcal{X}_α . \square

Corollary 6.2. *The coherent configuration \mathcal{X}_α is schurian and separable.*

Proof. By Corollary 4.6, the coherent configuration \mathcal{Y}_α is semiregular. By (46), this implies that \mathcal{X}_α is 1-regular. Thus, we are done by Theorem 2.3. \square

Let us prove that \mathcal{X} is separable. By Lemma 2.2 and Corollary 6.2, it suffices to verify that every algebraic isomorphism $\varphi : \mathcal{X} \rightarrow \mathcal{X}'$ can be extended to an algebraic isomorphism

$$(47) \quad \varphi_0 : \mathcal{X}_\alpha \rightarrow \mathcal{X}'_{\alpha'}.$$

To construct this extension, we note that by Theorem 5.6, given $\alpha \in \Omega'$, the mapping $\varphi_{\alpha, \alpha'} : r_{x,y} \mapsto r'_{x',y'}$, is an algebraic isomorphism between the saturated matching configurations \mathcal{M}_α and $\mathcal{M}'_{\alpha'}$. By Theorem 4.6, this algebraic isomorphism can be extended to the algebraic isomorphism

$$(48) \quad \psi_{\alpha, \alpha'} : \mathcal{Y}_\alpha \rightarrow \mathcal{Y}'_{\alpha'},$$

where \mathcal{Y}_α and $\mathcal{Y}'_{\alpha'}$ are the coherent closures of \mathcal{M}_α and $\mathcal{M}'_{\alpha'}$, respectively. It remains to note that by Lemma 6.1, the algebraic isomorphism (48) can be extended to the algebraic isomorphism (47) by setting

$$a^{\varphi_0} = \begin{cases} a^{\psi_{\alpha, \alpha'}} & \text{if } a \in M_\alpha, \\ r_{s^\varphi, t^\varphi} & \text{if } a \notin M_\alpha \text{ and } a = r_{s,t}, \end{cases}$$

here in view of formula (46), every basis relation of \mathcal{X}_α either belongs to M_α or is of the form $r_{s,t}$ with $s, t \in S$.

To prove that \mathcal{X} is schurian, fix $\alpha \in \Omega$ and set $\varphi = \text{id}_S$. By the result of the previous paragraph with $\mathcal{X}' = \mathcal{X}$ and $\alpha' \in \Omega$, the algebraic isomorphism φ can be extended to the algebraic isomorphism (47). Therefore the set

$$G_{\alpha \mapsto \alpha'} = \text{Iso}(\mathcal{X}_\alpha, \mathcal{X}_{\alpha'}, \varphi_0)$$

is not empty by Corollary 6.2. On the other hand, formula (8) shows this set is contained in the group $\text{Iso}(\mathcal{X}, \mathcal{X}, \text{id}_S) = \text{Aut}(\mathcal{X})$. Thus, the latter group is transitive, because it contains a transitive subgroup generated by the sets $G_{\alpha \mapsto \alpha'}$, $\alpha' \in \Omega$. Moreover,

$$\text{Orb}(\text{Aut}(\mathcal{X})_\alpha) = \{\alpha s : s \in S\}$$

by the second part of Lemma 6.1 and Corollary 6.2. Therefore, the coherent configuration associated with $\text{Aut}(\mathcal{X})$ coincides with \mathcal{X} , i.e., \mathcal{X} is schurian.

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